The Log-Approximate-Rank Conjecture is False

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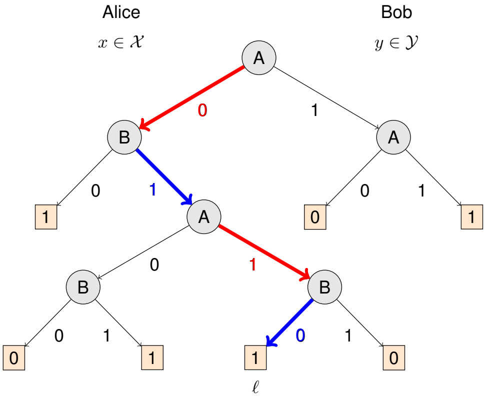
Communication Complexity

- How much do parties need to communicate in order to complete a task?
- Pops up everywhere. Streaming algorithms, extension polytopes, data structures and more.
- In this talk, we focus on two parties (Alice and Bob) computing a Boolean function.

A Communication Protocol



(x, y) is accepted

$\Leftrightarrow$

(x, y) reaches a 1-leaf.

Inputs that reach ℓ

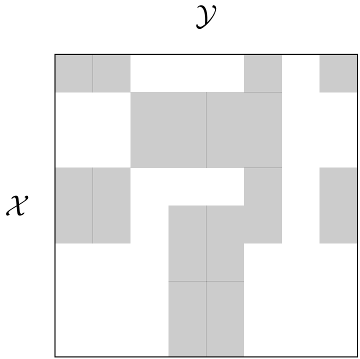
$=$

$\{x : x$ answers red$\}$

$\times$

$\{y : y$ answers blue$\}$.

Rank



Building the truth table for the function computed by the protocol:

Inputs that reach any 1 leaf form a rank $\leq 2^c$ matrix.

Cost c protocol for $F \implies M_F$ has rank $\leq 2^c$.

Protocol-Rank Equivalence?

Conjecture (Lovász Saks '88)

$$\exists \textit{ constant } \alpha \textit{ s.t. } D(F) \leq \log^{\alpha} \mathsf{rank}(F)$$

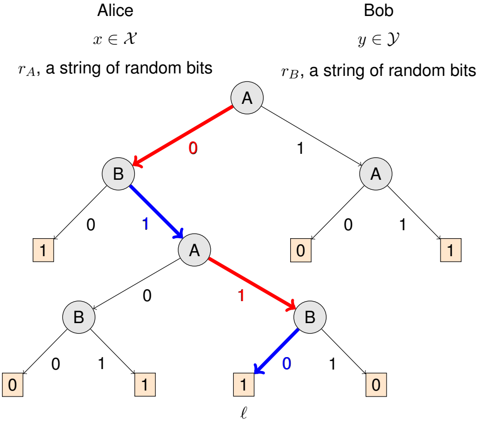
- Connects comm comp measure with algebraic measure. Known analogous connections have been useful.
- Has connections to graph colouring, low degree polynomials.

For: [Lovett '13] showed that $D(F) \lesssim O\left(\sqrt{\mathsf{rank}(F)}\right)$.

Against: [Göös Pitassi Watson '15] showed that $\alpha \geq 2$.

Fun fact: LRC is True if you restrict the rank decomposition to be nonnegative.

A Randomized Communication Protocol



$$\Pr[(x,y) \text{ is accepted}] = \Pr[(x,y) \text{ reaches a } 1\text{-leaf}].$$

$$\Pr[(x,y) \text{ reaches } \ell] = \Pr_{r_A}[x \text{ answers red}] \times \Pr_{r_B}[y \text{ answers blue}].$$

Small Approximate Rank

$\Pr_{r_B}[y$ answers blue$]$

$\Pr_{r_A}[x$ answers red$]$

	0	.5	0	.6
.5		.25		.3
.8		.4		.48
0				
0				

$\Pr[(x,y)$ reaches $\ell]$ is a rank 1 matrix.

$\Pr[(x,y)$ is accepted$]$ is a rank $\leq 2^c$ matrix.

1	1	0	0
0	1	0	0
0	0	1	0
0	0	0	1

M_F

Approx. Rank $\leq 2^c$

.8	.9	.1	.2
0	.9	.1	.1
0	.1	.8	0
.1	0	0	1

$M_{\text{Pr of accepting}}$

Rank $\leq 2^c$

$$\log \mathsf{rank}_{1/3}(F) \leq c.$$

Protocol-Rank Equivalence?

Conjecture (ForgeGod '05, Lee Shraibman '07)

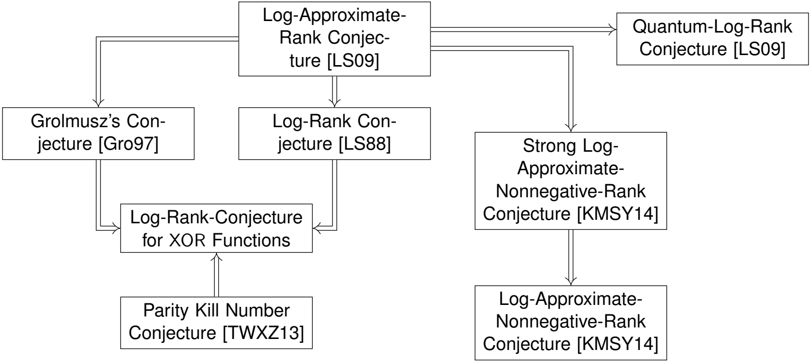
$\exists$ *constant* β *s.t.* $R(F) \leq \log^{\beta} \mathsf{rank}_{1/3}(F)$

Implies the LRC! [Gavinsky Lovett '13]

Set Disjointness shows that $\beta \geq 2$. [Kalyanasundaram Schnitger '92, Razborov '92]

[Göös Jayram Pitassi Watson '17] showed that $\beta \geq 4$.

The Web of Conjectures



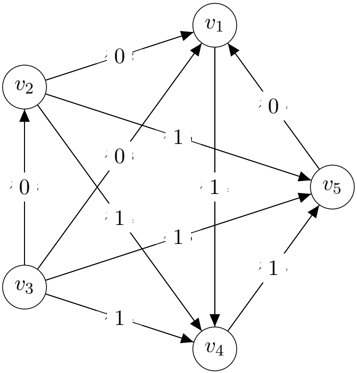
Protocol-Rank Non-Equivalence

Theorem (Chattopadhyay Mande S '19)

There is a function $F : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$ *such that* $\log \mathsf{rank}^+_{1/3}(F) \le O(\log n)$, *but* $R(F) \ge \Omega(\sqrt{n})$.

The Function

$$F := \mathsf{SINK} \circ \mathsf{XOR} : \{0,1\}^{\binom{m}{2}} \times \{0,1\}^{\binom{m}{2}} \to \{0,1\}$$



The input bits of SINK orient the edges of the complete graph.

$\mathsf{SINK}(z) = 1$ iff there is a sink in the directed graph G_z.

Alice

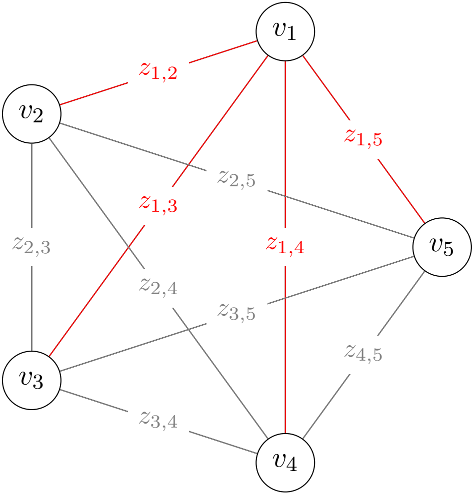
$x \in \{0,1\}^{\binom{m}{2}}$

$z = x \oplus y$

Bob

$y \in \{0,1\}^{\binom{m}{2}}$

Small Approximate Rank



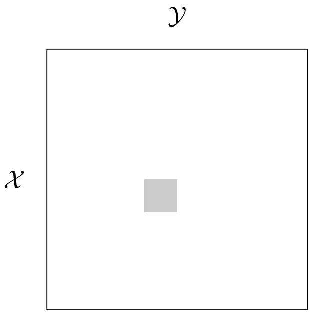
Whether or not v_1 is a sink is decided by the red variables, z_{v_1}.

v_1 is a sink iff $x_{v_1} = y_{v_1}$.

$M_{v_1 \text{ is a sink}}$ has small approximate rank.
(Because Equality has small approximate rank.)

$M_F = \sum M_{v_i \text{ is a sink}}$ has small approximate rank.

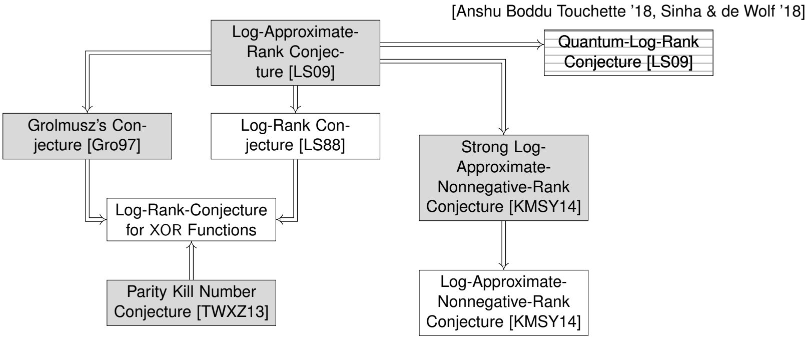
Randomized Communication Lower Bound



- A rectangle “biased” against v_1 being a sink must be small. (Follows from [Gavinsky ‘16].)
- Each additional vertex one “biases” against shrinks it further. (Near independence of sinks, Shearer’s lemma)
- A rectangle “biased” against sinks must be tiny.

Any randomized protocol for F must be costly.

Other Sunken Conjectures



So what now?

- ~~Quantum vs Log Approximate Rank?~~
- Can the Log Approximate Nonnegative Rank Conjecture be similarly refuted?
- What other functions refute the LARC?

Thank you for your attention.

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